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The Weighted L_p-Norms of Orthonormal Polynomials for Freud Weights

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Let $W := e^{-Q}$, where $Q: \mathbb{R} \to \mathbb{R}$ is even, continuous in $\mathbb{R}$, Q'' is continuous in $(0, \infty)$, and $Q' > 0$ in $(0, \infty)$, while for some $A, B > 1$,

$$A \leqslant \frac{d}{dx}(xQ'(x))/Q'(x) \leqslant B, \qquad x \in (0, \infty).$$

For example, $W(x) = \exp(-|x|^\alpha)$, $\alpha > 1$, satisfies these hypotheses. Let $p_n(W^2, x)$ denote the nth orthonormal polynomial for the weight W^2, $n \geqslant 1$, and let $a_n = a_n(Q)$ denote the nth Mhaskar–Rahmanov–Saff number for Q. We show that for $0 < p < \infty$, and $n \geqslant 2$,

$$\|p_n(W^2, \cdot)\, W(\cdot)\|_{L_p(\mathbb{R})} \sim a_n^{1/p - 1/2} \times \begin{cases} 1, & p < 4, \\ (\log n)^{1/4}, & p = 4, \\ (n^{-2/3})^{1/p - 1/4}, & p > 4. \end{cases}$$

The results are based on bounds for $p_n(W^2, \cdot)$ established recently by A. L. Levin and one of the authors. © 1994 Academic Press, Inc.

1. INTRODUCTION AND RESULTS

Let $W^2 := e^{-2Q}$, where $Q: \mathbb{R} \to \mathbb{R}$ is even, continuous, and is of "smooth polynomial growth" at infinity. Such a weight is often called a *Freud weight* [7, p. 83ff], and perhaps the archetypal example is

$$W_\alpha(x) := \exp(-|x|^\alpha), \qquad \alpha > 0.$$

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Corresponding to the weight W^2, we can define *orthonormal polynomials*

$$p_n(W^2, x) = \gamma_n x^n + \cdots, \qquad \gamma_n > 0,\ n \geqslant 0,$$

satisfying

$$\int_{-\infty}^{\infty} p_n(W^2, x)\, p_m(W^2, x)\, W^2(x)\, dx = \delta_{mn}, \qquad m, n \geqslant 0.$$

Recently, A. L. Levin and the first author [4] established bounds for $p_n(W^2, x)$ for a class of Freud weights that includes $\exp(-|x|^\alpha)$, $\alpha > 1$. Here we use these bounds, and other results in [3, 4] to estimate, both above and below, the L_p norms of $p_n(W^2, \cdot)\, W(\cdot)$. Such L_p estimates are useful in studying convergence of Lagrange interpolation and orthogonal expansions in weighted L_p norms.

To state our result, we need the *Mhaskar–Rahmanov–Saff* number a_u [5, 6], the positive root of the equation

$$u = \frac{2}{\pi} \int_0^1 a_u t Q'(a_u t)\, dt/\sqrt{1-t^2}, \qquad u > 0. \tag{1.1}$$

Under the conditions on Q below, which guarantee that $Q(s)$ and $Q'(s)$ increase strictly in $(0, \infty)$, a_u is uniquely defined, and increases with u. It grows roughly like $Q^{[-1]}(u)$, where $Q^{[-1]}$ denotes the inverse of Q on $(0, \infty)$.

We use $\sim$ in the following sense: If $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ are sequences of non-zero real numbers, we write

$$b_n \sim c_n,$$

if there exist C_1, $C_2 > 0$ independent of n, such that

$$C_1 \leqslant b_n/c_n \leqslant C_2, \qquad n \geqslant 1.$$

Following is our result:

THEOREM 1. *Let $W := e^{-Q}$, where $Q\colon \mathbb{R} \to \mathbb{R}$ is even and continuous in $\mathbb{R}$, Q'' is continuous in $(0, \infty)$, and $Q' > 0$ in $(0, \infty)$, while for some A, $B > 1$,*

$$A \leqslant \frac{d}{dx}(xQ'(x))/Q'(x) \leqslant B, \qquad x \in (0, \infty). \tag{1.2}$$

Let $a_n = a_n(Q)$ denote the nth Mhaskar–Rahmanov–Saff number for Q. Then, given, $0 < p < \infty$, we have for $n \geqslant 2$,

$$\|p_n(W^2, \cdot)\, W(\cdot)\|_{L_p(\mathbb{R})} \sim a_n^{1/p - 1/2} \times \begin{cases} 1, & p < 4, \\ (\log n)^{1/4}, & p = 4, \\ (n^{-2/3})^{1/p - 1/4}, & p > 4. \end{cases} \tag{1.3}$$

For $W(x) = W_\alpha(x) = \exp(-|x|^\alpha)$, $a_n \sim n^{1/\alpha}$, and so we deduce that

$$\|p_n(W^2, \cdot)\, W(\cdot)\|_{L_p(\mathbb{R})} \sim n^{(1/\alpha)(1/p - 1/2)} \times \begin{cases} 1, & p < 4, \\ (\log n)^{1/4}, & p = 4, \\ (n^{-2/3})^{1/p - 1/4}, & p > 4. \end{cases}$$

For the Hermite weight W_2, more precise asymptotics were given for $p = 4$ by G. Freud and G. Nemeth [2], in connection with a problem of Cieselski on the monotonicity properties of the weighted L_p norms of orthonormal Hermite functions.

We prove the theorem in the next section.

2. Proofs

Throughout, $C, C_1, C_2, \ldots$, denote positive constants independent of n, x, and $P \in \mathscr{P}_n$, where $\mathscr{P}_n$ denotes the set of real polynomials of degree $\leqslant n$. The same symbol does not necessarily denote the same constant in different occurrences.

As stated in the Introduction, our proofs depend on results from [4]. Throughout, we assume the hypotheses and notation of Theorem 1. First, we recall bounds from [4]: For simplicity, we write for $n \geqslant 1$,

$$p_n(x) := p_n(W^2, x).$$

Lemma 2.1. (a) *For $n \geqslant 1$,*

$$\sup_{x \in \mathbb{R}} |p_n(x)|\, W(x)\, |1 - |x|/a_n|^{1/4} \sim a_n^{-1/2}, \tag{2.1}$$

and

$$\sup_{x \in \mathbb{R}} |p_n(x)|\, W(x) \sim n^{1/6} a_n^{-1/2}, \tag{2.2}$$

(b) *For $n \geqslant 1$ and $x \in \mathbb{R}$,*

$$|p_n(x)|\, W(x) \leqslant C a_n^{-1/2} / [|1 - |x|/a_n|^{1/4} + n^{-1/6}]. \tag{2.3}$$

Proof. (a) These are Corollary 1.4 in [4].

(b) This follows directly from (2.1) and (2.2). ∎

Next, we recall a suitable infinite-finite range inequality:

LEMMA 2.2. *Let $0<p<\infty$. There exists $C>0$ such that for $n\geqslant 1$ and $P\in\mathscr{P}_n$,*

$$\|PW\|_{L_p(\mathbb{R})}\leqslant C\,\|PW\|_{L_p[-a_n,\,a_n]}. \tag{2.4}$$

Proof. This is a special case of Theorem 1.8 in [4]. ∎

We can now prove the upper bounds implicit in (1.3):

PROPOSITION 2.3. *Let $0<p<\infty$. There exists C_2 such that for $n\geqslant 2$,*

$$\|p_n(W^2,\cdot)\,W(\cdot)\|_{L_p(\mathbb{R})}\leqslant C_2 a_n^{1/p-1/2}\times\begin{cases}1, & p<4,\\ (\log n)^{1/4}, & p=4,\\ (n^{-2/3})^{1/p-1/4}, & p>4.\end{cases} \tag{2.5}$$

Proof. By Lemma 2.1(b), and Lemma 2.2,

$$\begin{aligned}\|p_n W\|^p_{L_p(\mathbb{R})} &\leqslant C_3\int_{-a_n}^{a_n} a_n^{-p/2}[|1-|x|/a_n|^{1/4}+n^{-1/6}]^{-p}\,dx\\ &=2C_3a_n^{-p/2}\int_0^{a_n}[(1-x/a_n)^{1/4}+n^{-1/6}]^{-p}\,dx\\ &=2C_3a_n^{1-p/2}n^{p/6-2/3}\int_0^{n^{2/3}}[t^{1/4}+1]^{-p}\,dt,\end{aligned} \tag{2.6}$$

by the substitution $1-x/a_n=n^{-2/3}t$. Here

$$\int_0^{n^{2/3}}[t^{1/4}+1]^{-p}\,dt\leqslant 1+\int_1^{n^{2/3}}t^{-p/4}\,dt\sim\begin{cases}(n^{2/3})^{1-p/4}, & p<4,\\ \log n, & p=4,\\ 1, & p>4.\end{cases} \tag{2.7}$$

Then (2.6) and (2.7) yield (2.5) on taking pth roots. ∎

In proving the lower bounds corresponding to Proposition 2.3, we need more results from [3, 4]. First, we recall a Markov–Bernstein inequality:

LEMMA 2.4. *For $n\geqslant 1$, $P\in\mathscr{P}_n$, and $x\in\mathbb{R}$,*

$$|(PW)'(x)|\leqslant C_1\frac{n}{a_n}\max\{n^{-2/3},\,1-|x|/a_n\}^{1/2}\,\|PW\|_{L_\infty(\mathbb{R})}. \tag{2.8}$$

Proof. See [3, Theorems 1.1, 1.3, pp. 1066–1067]. Note that our restriction $A > 1$ forces

$$\int_1^{Cn} \frac{ds}{Q^{[-1]}(s)} \sim \frac{n}{a_n}.$$

See (1.26)–(1.27) and Lemma 5.2(f) in [4]. ∎

We denote the *zeros* of $p_n(x)$ by

$$-\infty < x_{nn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n} < \infty.$$

The *fundamental polynomials of Lagrange interpolation are* $l_{jn} \in \mathscr{P}_{n-1}$ satisfying

$$l_{jn}(x_{kn}) = \delta_{jk}, \qquad 1 \leqslant j,\ k \leqslant n.$$

If we define the nth *Christoffel function* [7, p. 9]

$$\lambda_n(W^2, x) := \inf_{P \in \mathscr{P}_{n-1}} \int_{-\infty}^{\infty} (PW)^2\,(t)\,dt/P^2(x)$$
$$= 1 \Big/ \sum_{j=0}^{n-1} p_j^2(x),$$

then it is known that

$$l_{jn}(x) = \lambda_n(W^2, x_{jn})\,\frac{\gamma_{n-1}}{\gamma_n}\,p_{n-1}(x_{jn})\,\frac{p_n(x)}{x - x_{jn}}. \tag{2.9}$$

See, for example, [8, p. 6] or [1, p. 23].

LEMMA 2.5. (a) *For* $n \geqslant 1$ *and* $|x| \leqslant a_n$,

$$\lambda_n(W^2, x) \sim \frac{a_n}{n}\,W^2(x)\,\max\left\{n^{-2/3},\ 1 - \frac{|x|}{a_n}\right\}^{-1/2}. \tag{2.10}$$

(b) *For* $n \geqslant 1$,

$$|x_{1n}/a_n - 1| \leqslant Cn^{-2/3}, \tag{2.11}$$

and uniformly for $n \geqslant 3$ *and* $2 \leqslant j \leqslant n-1$,

$$x_{j-1,n} - x_{j+1,n} \sim \frac{a_n}{n}\,\max\{n^{-2/3},\ 1 - |x_{jn}|/a_n\}^{-1/2}. \tag{2.12}$$

(c) *Uniformly for* $n \geqslant 2$ *and* $1 \leqslant j \leqslant n-1$,

$$\max\{n^{-2/3}, 1-|x_{jn}|/a_n\} \sim \max\{n^{-2/3}, 1-|x_{j+1,n}|/a_n\}. \tag{2.13}$$

(d) *Uniformly for* $1 \leqslant j \leqslant n-1$ *and* $n \geqslant 2$,

$$|p_{n-1}(x_{jn})|\ W(x_{jn}) \sim a_n^{-1/2} \max\{n^{-2/3}, 1-|x_{jn}|/a_n\}^{1/4}. \tag{2.14}$$

(e) *For* $n \geqslant 1$, $1 \leqslant k \leqslant n$, *and* $x \in \mathbb{R}$,

$$|p_n(x)|\ W(x) \leqslant Cn/a_n^{3/2} \max\{n^{-2/3}, 1-|x|/a_n\}^{1/4}\ |x-x_{kn}|. \tag{2.15}$$

(f) *For* $n \geqslant 1$,

$$\gamma_{n-1}/\gamma_n \sim a_n. \tag{2.16}$$

Proof. (a) This is Theorem 1.1(a) in [4].

(b) This is Corollary 1.2 in [4].

(c) This is (11.10) in [4].

(d) This is Corollary 1.3 in [4].

(e) This is Theorem 12.3(a) in [4].

(f) This is Theorem 12.3(b) in [4]. ∎

LEMMA 2.6. (a) *Uniformly for* $n \geqslant 1$, $1 \leqslant j \leqslant n$, *and* $x \in \mathbb{R}$,

$$|l_{jn}(x)| \sim a_n^{3/2}/nW(x_{jn}) \max\{n^{-2/3}, 1-|x_{jn}|/a_n\}^{-1/4} \left|\frac{p_n(x)}{x-x_{jn}}\right|. \tag{2.17}$$

(b) *Uniformly for* $n \geqslant 1$, $1 \leqslant j \leqslant n$, *and* $x \in \mathbb{R}$,

$$|l_{jn}(x)|\ W^{-1}(x_{jn})\ W(x) \leqslant C. \tag{2.18}$$

(c) *There exists* $C_1 > 0$, *such that uniformly for* $n \geqslant 1$, $1 \leqslant j \leqslant n$, *and*

$$|x-x_{jn}| \leqslant C_1 \frac{a_n}{n} \max\{n^{-2/3}, 1-|x_{jn}|/a_n\}^{-1/2}, \tag{2.19}$$

we have

$$|p_n(x)|\ W(x) \sim n/a_n^{3/2} \max\{n^{-2/3}, 1-|x_{jn}|/a_n\}^{1/4}\ |x-x_{jn}|. \tag{2.20}$$

Proof. (a) This is an immediate consequence of (2.9), (2.10), (2.14), and (2.16).

(b) By (2.17) and (2.15), we have

$$|l_{jn}(x)|\ W^{-1}(x_{jn})\ W(x) \leqslant C\left(\frac{\max\{n^{-2/3}, 1-|x|/a_n\}}{\max\{n^{-2/3}, 1-|x_{jn}|/a_n\}}\right)^{1/4}.$$

If for some fixed $\lambda > 0$,

$$\max\{n^{-2/3}, 1-|x|/a_n\} \leqslant \lambda \max\{n^{-2/3}, 1-|x_{jn}|/a_n\}, \tag{2.21}$$

then we obtain (2.18). If we set

$$x_{1-l,n} := x_{1n} + la_n n^{-2/3}; \qquad x_{n+l,n} := x_{nn} - la_n n^{-2/3},$$

$l = 1, 2$, then (2.13) shows that (2.21) is true for $x \in (x_{j-2,n}, x_{j+2,n})$, with a suitably large λ. On the other hand, if (2.21) is not true, so that $x \notin (x_{j-2,n}, x_{j+2,n})$, then (2.3) and (2.17) show that

$$\begin{aligned}
&|l_{jn}(x)|\ W^{-1}(x_{jn})\ W(x)\\
&\quad\leqslant C_2 a_n^{3/2}/n \max\{n^{-2/3}, 1-|x_{jn}|/a_n\}^{-1/4}\\
&\qquad\times a_n^{-1/2}[\,|1-|x|/a_n|^{1/4} + n^{-1/6}]^{-1}\ |x_{j\pm 2,n} - x_{jn}|^{-1}\\
&\quad\leqslant C_3 \max\{n^{-2/3}, 1-|x_{jn}|/a_n\}^{1/4}\ [\,|1-|x|/a_n|^{1/4} + n^{-1/6}]^{-1}\\
&\qquad\qquad\text{(by (2.12) and (2.13))}\\
&\quad\leqslant C_4\left(\frac{\max\{n^{-2/3}, 1-|x_{jn}|/a_n\}}{\max\{n^{-2/3}, 1-|x|/a_n\}}\right)^{1/4} \leqslant C_4 \lambda^{-1/4},
\end{aligned}$$

as (2.21) does not hold. So we still have (2.18). Thus (2.18) holds for $x \in \mathbb{R}$.

(c) Consider the polynomial

$$\tau_{jn}(x) := l_{jn}(x)\ W^{-1}(x_{jn}).$$

We have

$$(\tau_{jn} W)(x_{jn}) = 1,$$

and according to (b) of this lemma,

$$\|\tau_{jn} W\|_{L_\infty(\mathbb{R})} \leqslant C,$$

with C independent of j and n. Now let $\eta > 0$ be fixed, and let

$$\varepsilon := \varepsilon(j, n) := \eta\, \frac{a_n}{n} \max\{n^{-2/3}, 1-|x_{jn}|/a_n\}^{-1/2}. \tag{2.22}$$

Let $x_{1-l,n}$ and $x_{n+l,n}$, $l=1,2$, be as defined in (b). Now if η is small enough (the upper bound on η being independent of j, n), (2.11) and (2.12) show that uniformly for $1 \leqslant j \leqslant n$,

$$(x_{jn}-\varepsilon,\ x_{jn}+\varepsilon) \subset (x_{j-2,n},\ x_{j+2,n}). \tag{2.23}$$

Furthermore, for $s \in (x_{jn}-\varepsilon,\ x_{jn}+\varepsilon)$, (2.13) and the Markov–Bernstein inequality Lemma 2.4 show that

$$|(\tau_{jn} W)'(s)| \leqslant C_1 \frac{n}{a_n} \max\{n^{-2/3},\ 1-|x_{jn}|/a_n\}^{1/2}.$$

Hence, if $t \in (x_{jn}-\varepsilon,\ x_{jn}+\varepsilon)$, we have for some s between t and x_{jn},

$$\begin{aligned} |\tau_{jn} W|\,(t) &= |(\tau_{jn} W)(x_{jn}) + (\tau_{jn} W)'(s)(t-x_{jn})| \\ &\geqslant 1 - C_1 \frac{n}{a_n} \max\{n^{-2/3},\ 1-|x_{jn}|/a_n\}^{1/2}\, \varepsilon \\ &= 1 - C_1\eta \geqslant 1/2, \end{aligned}$$

if η in the choice (2.22) of ε is small enough. Thus

$$|\tau_{jn} W|\,(t) \sim 1, \qquad t \in (x_{jn}-\varepsilon,\ x_{jn}+\varepsilon),$$

and recalling (2.17) and the definition of τ_{jn}, we have (2.20). ∎

Proof of Theorem 1. Fix $1 \leqslant j \leqslant n$, and with C_1 as in (2.19), let

$$\varepsilon := C_1 \frac{a_n}{n} \max\{n^{-2/3},\ 1-|x_{jn}|/a_n\}^{-1/2}.$$

Then, recalling (2.23), we have

$$\begin{aligned} &\int_{x_{j-2,n}}^{x_{j+2,n}} |p_n W|\,(x)^p\, dx \\ &\quad \geqslant C_2 \int_{x_{jn}-\varepsilon}^{x_{jn}+\varepsilon} (n/a_n^{3/2} \max\{n^{-2/3},\ 1-|x_{jn}|/a_n\}^{1/4})^p\, |x-x_{jn}|^p\, dx \\ &\qquad\qquad \text{(by (2.20))} \\ &\quad \geqslant C_3 (n/a_n^{3/2} \max\{n^{-2/3},\ 1-|x_{jn}|/a_n\}^{1/4})^p\, \varepsilon^{p+1} \\ &\quad \geqslant C_3 a_n^{1-p/2}/n \max\{n^{-2/3},\ 1-|x_{jn}|/a_n\}^{-p/4-1/2} \\ &\quad \geqslant C_4 a_n^{-p/2} (x_{j-2,n}-x_{j+2,n}) \max\{n^{-2/3},\ 1-|x_{jn}|/a_n\}^{-p/4} \\ &\qquad\qquad \text{(by (2.12) and (2.13))} \\ &\quad \geqslant C_5 a_n^{-p/2} \int_{x_{j-2,n}}^{x_{j+2,n}} \max\{n^{-2/3},\ 1-|t|/a_n\}^{-p/4}\, dt, \end{aligned}$$

in view of (2.13). Summing, we have

$$\int_{-\infty}^{\infty} |p_n W|\,(x)^p\,dx$$

$$\geqslant C_5 a_n^{-p/2} \int_{x_{nn}}^{x_{1n}} \max\{n^{-2/3}, 1-|t|/a_n\}^{-p/4}\,dt$$

$$= C_5 a_n^{1-p/2} \int_{x_{nn}/a_n}^{x_{1n}/a_n} \max\{n^{-2/3}, 1-|s|\}^{-p/4}\,ds$$

$$\geqslant C_6 a_n^{1-p/2} \int_{-1+C_7 n^{-2/3}}^{1-C_7 n^{-2/3}} (1-|s|)^{-p/4}\,ds \qquad \text{(by (2.11))}$$

$$\geqslant C_7 a_n^{1-p/2} \times \begin{cases} 1, & p \leqslant 4, \\ \log n, & p = 4, \\ (n^{-2/3})^{1-p/4}, & p > 4. \end{cases}$$

Hence

$$\|p_n W\|_{L_p(\mathbb{R})} \geqslant C_8 a_n^{1/p-1/2} \times \begin{cases} 1, & p < 4, \\ (\log n)^{1/4}, & p = 4, \\ (n^{-2/3})^{1/p-1/4}, & p > 4. \end{cases}$$

Together with Proposition 2.3, this yields the result. ∎

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